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Uniform embeddability and exactness of free products

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Abstract

Let A and B be countable discrete groups and let $\Gamma = A * B$ be their free product. We show that if both A and B are uniformly embeddable in a Hilbert space then so is Γ. We give two different proofs: the first directly constructs a uniform embedding of Γ from uniform embeddings of A and B; the second works without change to show that if both A and B are exact then so is Γ.

1. Introduction

The concept of uniform embedding into Hilbert space was introduced by Gromov [Gro93]. It plays an important role in the study of the Novikov higher signature conjecture [FRR95,STY00,Yu00].

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Let X be a countable discrete metric space and let d denote its metric; let $\mathscr{H}$ be a separable and infinite-dimensional Hilbert space. A map $F : X \to \mathscr{H}$ is a *uniform embedding* [Gro93] if there exist non-decreasing functions ρ_1 and ρ_2 from $\mathbb{R}_+ = [0, \infty)$ to $\mathbb{R}$ such that

(1) $\rho_1(d(x,y)) \leqslant ||F(x) - F(y)|| \leqslant \rho_2(d(x,y))$, for all $x, y \in X$, and
(2) $\lim_{t\to+\infty} \rho_i(t) = +\infty$, for $i = 1, 2$.

The discrete metric space X is *uniformly embeddable* if it admits exists a uniform embedding.

A countable discrete group Γ is *exact* if the functor given by the reduced crossed product with Γ converts a short exact sequence of Γ-C^*-algebras into a short exact sequence of C^*-algebras. Equivalently, the functor given by spatial tensor product with the reduced C^*-algebra of Γ converts one short exact sequence of C^*-algebras into another. The class of exact groups is closed under a number of operations [KW99], including the formation of free products (both with and without amalgam) [Dyk99,Dyk00].

Theorem. *Let A and B be countable discrete groups and let $\Gamma = A * B$ be their free product. If both A and B are uniformly embeddable in a Hilbert space then so is Γ; if both A and B are exact then so is Γ.*

We discuss the theorem from several different perspectives. We first give a direct proof of the uniform embeddability result; given uniform embeddings of the factors A and B we explicitly construct the required uniform embedding of Γ. Subsequently, relying on recent characterizations of uniform embeddability and exactness [GK02,Oza00], we give a unified proof of both statements in the theorem. Although the statement concerning exactness is known, our proof is more elementary than and unrelated to the original proof of Dykema and its successive refinements [Dyk99,Dyk00].

The theorem leaves open the question of whether a free product $A *_G B$ with non-trivial amalgamation is uniformly embeddable in a Hilbert space if each of the factors A and B are. Indeed, this is proven by Dadarlat–Guentner [DG02].

2. Preliminaries

We recall some elementary facts about length functions and metrics on discrete groups and free products. An integer valued function ℓ on a group G is a *length function* if

(1) $\ell(g) = \ell(g^{-1}) \geqslant 0$ for all $g \in G$,
(2) $\ell(e) = 0$,
(3) $\ell(gh) \leqslant \ell(g) + \ell(h)$, for all g, $h \in G$.

A length function ℓ is *non-degenerate* if $\ell(g) = 0$ implies that $g = e$. For any non-degenerate length function ℓ on G, we define the *associated metric* d on G by $d(g,h) = \ell(g^{-1}h)$, for all g, $h \in G$.

A length function ℓ is *proper* if $\ell^{-1}(S)$ is a finite set for every finite subset S of $\mathbb{N}$. A group G admits a proper and non-degenerate length function if and only if it is countable.

Remark. Although we do not require it, we point out that uniform embeddability of a countable discrete group G, equipped with a metric associated to a proper and non-degenerate length function, does not depend on the choice of the length function.

Next, we recall some elementary facts about free products. Let A and B be countable discrete groups and let $\Gamma = A * B$ be their free product. Every element $g \in \Gamma$ is uniquely expressed in normal form as a reduced word $g = x_1 \dots x_p$, where it is understood that $x_i \in A \cup B$, $x_i \neq e$ and if $x_i \in A$ (or B) then $x_{i+1} \in B$ (or A), as appropriate.

Let ℓ_A and ℓ_B be proper non-degenerate integer valued length functions on A and B, respectively. Define an integer valued function $\ell = \ell_\Gamma$ on Γ by

$$\ell_\Gamma(g) = \sum_1^n \ell_A(a_i) + \sum_1^n \ell_B(b_i),$$

where we have written $g = a_1 b_1 \dots a_n b_n$ as a product without cancellation and $a_i \in A$, $b_j \in B$. It is easy to see that ℓ_Γ is a proper non-degenerate length function. Let d_Γ be the metric associated to ℓ_Γ. Quite explicitly, if g, $g' \in \Gamma$ we write g and g' as products without cancellation,

$$g = hxx_1 \dots x_n,$$

$$g' = hx'x'_1 \dots x'_m, \tag{1}$$

where h is the common part of g and g', $x \neq x' \in A$ (or B) and $x_1, \dots, x_n$ are alternately elements of B and A (or A and B) and similarly for $x'_1, \dots, x'_m$, as is consistent with normal form expressions. (We allow the degenerate cases (i) $h = e$, $x \neq x'$ and (ii) $h = x = x' = e$; observe that in (ii) one of x_1 and x'_1 is from A whereas the other is from B.) Having done so, and with the convention that empty sums are zero, we obtain

$$d_\Gamma(g, g') = \sum_1^n \ell_{A,B}(x_i) + d_{A,B}(x, x') + \sum_1^m \ell_{A,B}(x'_j), \tag{2}$$

where we have written $\ell_{A,B}$ to mean ℓ_A or ℓ_B as appropriate, and similarly for $d_{A,B}$. (In the degenerate case, $h = x = x' = e$ the middle term does not appear.) Observe that, since our length functions are integer valued, the number of non-zero terms in this expression is not greater than $d_\Gamma(g, g')$.

3. Construction of an embedding

Given uniform embeddings of A and B, we explicitly construct a uniform embedding of their free product Γ. For the construction we require two lemmas.

Lemma 1. *If a countable discrete metric space X is uniformly embeddable into Hilbert space, then there exists a uniform embedding $F : X \to \mathcal{H}$ and, for $i = 1, 2$, non-decreasing functions $\rho_i : \mathbb{R}_+ \to \mathbb{R}$ such that*

(1) $\rho_1(d(x,y)) \leqslant ||F(x) - F(y)|| \leqslant \rho_2(d(x,y))$, *for all* $x, y \in X$,
(2) $\lim_{t\to+\infty} \rho_i(t) = +\infty$ *for* $i = 1, 2$,
(3) $\rho_1(1) \geqslant 1$.

Proof. Let $\tilde{F}: X \to \mathcal{H}$ be a uniform embedding and, for $i = 1, 2$, let $\tilde{\rho}_i$ be non-decreasing functions on $\mathbb{R}_+$ satisfying

(1) $\tilde{\rho}_1(d(x,y)) \leqslant ||\tilde{F}(x) - \tilde{F}(y)|| \leqslant \tilde{\rho}_2(d(x,y))$ for all $x, y \in X$,
(2) $\lim_{t\to+\infty} \tilde{\rho}_i(t) = +\infty$.

We define another uniform embedding as follows

$$F : X \to H \oplus l^2(X), \quad F(x) = \tilde{F}(x) \oplus \delta_x,$$

where δ_x is the Dirac function at the point x. Let

$$\rho_1(t) = \begin{cases} 0, & \text{when } t = 0, \\ \sqrt{(\tilde{\rho}_1(t))^2 + 1}, & \text{when } t > 0 \end{cases}$$

and

$$\rho_2(t) = \sqrt{(\tilde{\rho}_2(t))^2 + 2}.$$

It is easy to verify that ρ_1 and ρ_2 satisfy the desired conditions. $\square$

Lemma 2. *If $\rho(t)$ is non-decreasing function satisfying $\lim_{t\to+\infty} \rho(t) = +\infty$ and $\rho(1) \geqslant 1$, then there exists a non-decreasing function $\tilde{\rho}(t)$ such that $\lim_{t\to+\infty} \tilde{\rho}(t) = +\infty$ and such that for all $n \in \mathbb{N}$ and $(t_i)_1^n \in \mathbb{N}$ we have*

$$\sum_{i=1}^{n} \rho(t_i) \geqslant \tilde{\rho}\left(\sum_{i=1}^{n} t_i\right).$$

Proof. Define $\tilde{\rho}(t) = \min\left(\frac{\sqrt{t}}{2}, \rho\left(\frac{\sqrt{t}}{2}\right)\right)$. The only non-trivial condition to check is that $\tilde{\rho}$ satisfies the above inequality. Let n and t_i be as in the statement; denote $N = \sum_{i=1}^{n} t_i$.

Case $n \leqslant \frac{\sqrt{N}}{2}$: In this case, there exists a natural number $i_0 \leqslant n$ such that $t_{i_0} > \frac{\sqrt{N}}{2}$; otherwise

$$\sum_{i=1}^{n} t_i \leqslant \frac{\sqrt{N}}{2}\frac{\sqrt{N}}{2} = \frac{N}{4} < N,$$

contradicting the assumption. Therefore,

$$\sum_{i=1}^{n} \rho(t_i) \geqslant \rho(t_{i_0}) \geqslant \rho\left(\frac{\sqrt{N}}{2}\right) = \rho\left(\frac{\sqrt{\sum_{i=1}^{n} t_i}}{2}\right) \geqslant \tilde{\rho}\left(\sum_{i=1}^{n} t_i\right).$$

Case $n > \frac{\sqrt{N}}{2}$: In this case, we have

$$\sum_{i=1}^{n} \rho(t_i) > \frac{\sqrt{N}}{2} = \frac{\sqrt{\sum_{i=1}^{n} t_i}}{2} \geqslant \tilde{\rho}\left(\sum_{i=1}^{n} t_i\right),$$

since $\rho(1) \geqslant 1$ and $\sum_{i=1}^{n} t_i = N$. $\square$

Proof of the Theorem (Embeddability). Let A and B be countable discrete groups. Equip A, B and Γ with proper non-degenerate length functions as described above. Denote by W_A the set of those elements of Γ whose expression as a reduced word begins with A; similarly W_B. By convention e is an element of both W_A and W_B. Notice that the union $W_A \cup W_B$ is Γ and that the intersection $W_A \cap W_B$ is $\{e\}$.

Assume that A and B are uniformly embeddable and let $F_A : A \to \mathscr{H}$ and $F_B : B \to \mathscr{H}_B$ be uniform embeddings. By Lemma 1, we can assume that there are non-decreasing functions ρ_1 and ρ_2 with $\lim_{t\to+\infty} \rho_i(t) = +\infty$, $i = 1, 2$, such that $\rho_1(1) \geqslant 1$ and such that

$$\begin{aligned} \rho_1(d(a,a')) &\leqslant \|F_A(a) - F_A(a')\| \leqslant \rho_2(d(a,a')), \\ \rho_1(d(b,b')) &\leqslant \|F_B(b) - F_B(b')\| \leqslant \rho_2(d(b,b')), \end{aligned}$$

for all $a, a' \in A$ and $b, b' \in B$. Adjusting each of F_A and F_B by a unitary isomorphism if necessary we further assume that $\mathscr{H}_B = \mathscr{H}$, and that $F_A(e) = F_B(e) = 0$.

Define a new Hilbert space $\mathscr{H}_\Gamma$ by

$$\mathscr{H}_\Gamma = \left(\bigoplus_{W_A} \mathscr{H}\right) \oplus \left(\bigoplus_{W_B} \mathscr{H}\right).$$

Observe that $\mathscr{H}_\Gamma$ is not quite the direct sum of copies of $\mathscr{H}$ indexed by the elements of Γ; an element of $\mathscr{H}_\Gamma$ has a component in $\mathscr{H}$ at every element of the *disjoint* union of W_A and W_B. We write an element $x \in \mathscr{H}_\Gamma$ as $x = x_A \oplus x_B$ where $x_A = \bigoplus_{h \in W_A} x_h$ and $x_B = \bigoplus_{h \in W_B} x_h$, and record the fact that for $x, y \in \mathscr{H}_\Gamma$ we have

$$\|x - y\|^2 = \sum_{h \in W_A} \|x_h - y_h\|^2 + \sum_{h \in W_B} \|x_h - y_h\|^2.$$

Next define a uniform embedding $F:\Gamma\to\mathscr{H}_\Gamma$. We define $F(e)=0$. If g is a non-identity element of Γ write g as a reduced word $g=x_1\dots x_p$, where the $x_i\neq e$ are alternately elements of A and B; for definiteness in the subsequent formula we assume that $x_1\in A$. In this case, the components of $F(g)$ at elements of W_B will be zero; its components at elements of W_A are defined, with the convention that an empty product is e, as follows:

$$F(g)_A=\bigoplus_{h\in W_A}F(g)_h,\quad F(g)_h=\begin{cases}F_A(x_{2k+1}), & \exists k\geqslant 0 \text{ such that } h=x_1\dots x_{2k},\\ F_B(x_{2k}), & \exists k\geqslant 1 \text{ such that } h=x_1\dots x_{2k-1},\\ 0, & \text{otherwise.}\end{cases}$$

In particular, $F(g)_e=F_A(x_1)$, $F(g)_{x_1}=F_B(x_2)$, $F(g)_{x_1x_2}=F_A(x_3),\dots$. Note that the component of $F(g)$ at $h\in W_A$ is non-zero precisely when $h\neq g$ and the reduced word of h is an initial segment (possibly empty) of the reduced word of g. A similar formula is used and similar remarks apply when the reduced word expression of g begins with an element of B.

It remains to show that F is indeed a uniform embedding. Let g, $g'\in\Gamma$ and write $g=hxx_1\dots x_n$ and $g'=hx'x_1'\dots x_m'$ as products without cancellation as in (1). We have, with the convention that empty sums are zero:

$$||F(g)-F(g')||^2=\sum_1^n||F_{A,B}(x_i)||^2+||F_{A,B}(x)-F_{A,B}(x')||^2+\sum_1^m||F_{A,B}(x_j')||^2,\quad (3)$$

where we have written $F_{A,B}$ to mean either F_A or F_B as appropriate. (In the case $h=x=x'=e$ the middle term does not appear.) Considering this expression we bound $||F(g)-F(g')||^2$ above by

$$\begin{aligned}||F(g)-F(g')||^2&\leqslant\sum_1^n\rho_2^2(\ell_{A,B}(x_i))+\rho_2^2(d_{A,B}(x,x'))+\sum_1^m\rho_2^2(\ell_{A,B}(x_j'))\\&\leqslant d_\Gamma(g,g')\rho_2^2(d_\Gamma(g,g')),\end{aligned}$$

recalling that, since our length functions are integer valued, the number of non-zero terms on the right is not greater than $d_\Gamma(g,g')$. Defining $\eta_2(t)=\sqrt{t}\rho_2(t)$ we therefore have

$$||F(g)-F(g')||\leqslant\eta_2(d_\Gamma(g,g')).$$

Again considering (3) we bound $||F(g)-F(g')||^2$ below by

$$||F(g)-F(g')||^2\geqslant\sum_1^n\rho_1^2(\ell_{A,B}(x_i))+\rho_1^2(d_{A,B}(x,x'))+\sum_1^m\rho_1^2(\ell_{A,B}(x_j')).$$

Define $\eta_1(t) = \sqrt{\tilde{\rho}(t)}$, where $\tilde{\rho}$ is as in Lemma 2 applied to $\rho = \rho_1^2$. By Lemma 2, we obtain

$$||F(g) - F(g')|| \geqslant \eta_1(d_\Gamma(g, g')). \qquad \square$$

4. Uniform embeddability and exactness

We will use the following characterizations of uniform embeddability and exactness [GK02,Oza00]. A countable discrete group Γ is uniformly embeddable if and only if for every $\varepsilon > 0$ and every $C > 0$ there exists a Hilbert space valued function $\xi : \Gamma \to \mathscr{H}$, $(\xi_a)_{a \in \Gamma}$ such that $||\xi_a|| = 1$ and

(1) $||\xi_a - \xi_b|| < \varepsilon$ if $d(a, b) \leqslant C$;
(2) for all $\hat{\varepsilon} > 0$ there exists $R > 0$ such that $|\langle \xi_a, \xi_b \rangle| < \hat{\varepsilon}$ if $d(a, b) \geqslant R$.

A countable discrete group Γ is exact if and only if for every $\varepsilon > 0$ and every $C > 0$ there exists a Hilbert space valued function $\xi : \Gamma \to \mathscr{H}$, $(\xi_a)_{a \in \Gamma}$ such that $||\xi_a|| = 1$ and

(1) $||\xi_a - \xi_b|| < \varepsilon$ if $d(a, b) \leqslant C$;
(2) there exists $R > 0$ such that $\langle \xi_a, \xi_b \rangle = 0$ if $d(a, b) \geqslant R$.

We refer to conditions (1) and (2) as the *convergence* and *support conditions*, respectively.

From the perspective of these conditions, uniform embeddability and exactness appear to be very similar. Indeed, based on these characterizations we give below a unified proof of both statements in the theorem. Further, as one might expect, the class of uniformly embeddable groups shares, in nearly every case, the closure properties of the class of exact groups; a systematic treatment of these ideas appears elsewhere [DG02].

In the proof of the theorem, we require one preliminary result. A *tree* T consists of two sets, a set V of vertices and a set E of edges, together with two *endpoint maps* $E \to V$ associating to each edge its endpoints. Every two vertices are connected by a unique geodesic edge path, that is, one without backtracking. It is convenient to define a metric on E by

$$d_T(e,f) = \text{the number of vertices on the unique path in } T \text{ from } e \text{ to } f.$$

Observe that $d_T(e,f)$ is simply half the number of edges on the unique path from e to f viewed as vertices in the first barycentric subdivision of T. The following lemma is a straightforward adaptation to the present situation of a well-known construction (compare [Yu00]).

Lemma 3. *Let T be a tree. For every $N \in \mathbb{N}$, there exists a Hilbert space valued function $\tau_N : E \to \mathscr{H}$ such that $||\tau_{N,e}|| = 1$, and*

(1) *if $d_T(e,f) \geqslant 2N$ then $\langle \tau_{N,e}, \tau_{N,f} \rangle = 0$,*
(2) $||\tau_{N,e} - \tau_{N,f}||^2 \leqslant 2d_T(e,f)/N$.

Proof. Let $\mathcal{H} = l^2(V) \oplus l^2(\mathbb{N}_+)$ and let $\hat{v}$ be a fixed vertex in V. Define

$$\tau_{N,e} = \begin{cases} \frac{1}{\sqrt{N}}(\delta_{v_{e,1}} + \cdots + \delta_{v_{e,N}}) \oplus 0, & N \leqslant k, \\ \frac{1}{\sqrt{N}}\{(\delta_{v_{e,1}} + \cdots + \delta_{v_{e,k}}) \oplus (\delta_1 + \cdots + \delta_{N-k})\}, & N > k, \end{cases}$$

where $v_{e,1}, \ldots, v_{e,k} = \hat{v}$ are the vertices along the unique path from e to $\hat{v}$. The assertions are easily verified. $\square$

Proof of the Theorem (Embeddability and exactness). We concentrate on the statement about uniform embeddability; nevertheless, our proof applies equally well to the statement concerning exactness, on which we comment briefly at the end. Let $\varepsilon > 0$ and $C > 0$ be given. As in the characterization of uniform embeddability above obtain a Hilbert space valued function $\alpha : A \to \mathcal{H}$ such that $||\alpha_a|| = 1$, and

(1) $||\alpha_a - \alpha_{a'}|| < \varepsilon/2C$, if $\ell_A(a^{-1}a') \leqslant C$,
(2) $\forall \hat{\varepsilon} > 0\ \exists R > 0$ such that $|\langle \alpha_a, \alpha_{a'} \rangle| < \hat{\varepsilon}$, if $\ell_A(a^{-1}a') \geqslant R$.

Similarly, obtain a Hilbert space valued function $\beta : B \to \mathcal{H}_B$. It is convenient to assume, as we may by applying an appropriate unitary operator $\mathcal{H}_B \to \mathcal{H}$, that $\beta : B \to \mathcal{H}$ satisfies $\alpha_e = \beta_e$ and

(1) $||\beta_b - \beta_{b'}|| < \varepsilon/2C$, if $\ell_B(b^{-1}b') \leqslant C$,
(2) $\forall \hat{\varepsilon} > 0\ \exists R > 0\ |\langle \beta_b, \beta_{b'} \rangle| < \hat{\varepsilon}$, if $\ell_B(b^{-1}b') \geqslant R$.

We view $\mathcal{H}$ as a pointed Hilbert space with distinguished vector $\omega = \alpha_e = \beta_e$.

Using the Bass–Serre tree T_Γ of Γ we define a single Hilbert space $\mathcal{H}_\Gamma$. Recall that the vertex set of T_Γ is $V_\Gamma = \Gamma/A \cup \Gamma/B$, and that its edge set is $E_\Gamma = \Gamma$. Associate to each vertex v the Hilbert space $\mathcal{H}_v = \mathcal{H}$ and define

$$\mathcal{H}_\Gamma = \lim_{\substack{F \subset V_\Gamma \\ \text{finite}}} \bigotimes_{v \in F} \mathcal{H}_v.$$

A few remarks concerning this definition are in order. First, if $F \subset G$ are finite subsets of V_Γ the map $\otimes_{v \in F} \mathcal{H}_v \to \otimes_{v \in G} \mathcal{H}_v$ is given by inserting the distinguished vector ω for those $v \in G \backslash F$. Second, these maps are isometries so that the algebraic direct limit of the $\otimes_{v \in F} \mathcal{H}_v$ is an inner product space in a natural way. Finally, $\mathcal{H}_\Gamma$ is obtained by completion. For notational convenience, we shall regard the formal infinite tensor product $\rho = \otimes_{v \in V_\Gamma} \rho_v$, where all but finitely many of the $\rho_v = \omega$, as an element of $\mathcal{H}_\Gamma$. Such elements span a dense linear subspace of $\mathcal{H}_\Gamma$. If σ is another such element then we have

$$\langle \rho, \sigma \rangle = \prod_{v \in V_\Gamma} \langle \rho_v, \sigma_v \rangle_{\mathscr{H}_v}, \quad \|\rho - \sigma\| \leqslant \sum_{v \in V_\Gamma} \|\rho_v - \sigma_v\|_{\mathscr{H}_v},$$

where in the expression for the norm we assume that the components of ρ and σ have norm not greater than 1. Observe that all but finitely many of the terms in the infinite product and sum are 1 and 0, respectively.

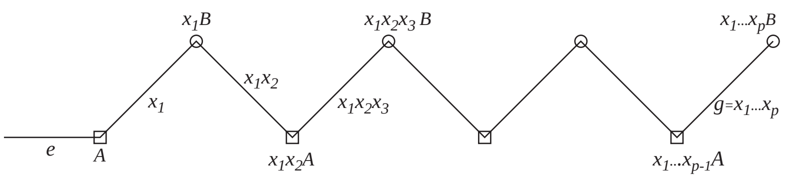
Define $\gamma : \Gamma \to \mathscr{H}_\Gamma$ by the formal infinite tensor product expression

$$\gamma_g = \bigotimes_{v \in V_\Gamma} \gamma_{g,v},$$

where the component $\gamma_{g,v}$ of γ_g at the vertex v is defined recursively: $\gamma_{e,v} = \omega$, and

$$\gamma_{gx,v} = \begin{cases} \gamma_{g,v}, & v \neq gA, \\ \alpha_x, & v = gA, \end{cases}$$

where gx is a product without cancellation and $e \neq x \in A$. A similar formula is used when $e \neq x \in B$. Equivalently, consider the normal form expression $g = x_1 \ldots x_p$, and assume for definiteness that $x_1,\, x_p \in A$. In this case, the recursive expression for γ_g is best understood by considering the following portion of the Bass–Serre tree for Γ:



For vertices *on* the path from e to g, the components of γ_g are given according to

$$\alpha_{x_1} \otimes \beta_{x_2} \otimes \alpha_{x_3} \otimes \cdots \otimes \alpha_{x_p} \in \mathscr{H}_A \otimes \mathscr{H}_{x_1B} \otimes \mathscr{H}_{x_1x_2A} \otimes \cdots \otimes \mathscr{H}_{x_1 \ldots x_{p-1}A}.$$

For vertices *not* on this path, we have $\gamma_{g,v} = \omega$. Again, similar formulas hold in the cases when one or both $x_1,\, x_p \in B$.

The function γ will in general *not* satisfy the support condition. In order to remedy this, let $N \geqslant 8C\varepsilon^{-2}$ and obtain $\tau = \tau_N : \Gamma \to \mathscr{H}$ as in Lemma 3. Define

$$\xi : \Gamma \to \mathscr{H}_\Gamma \otimes \mathscr{H}, \quad \xi_g = \gamma_g \otimes \tau_g.$$

It is clear that $\|\xi_g\| = 1$. We verify the support and convergence properties.

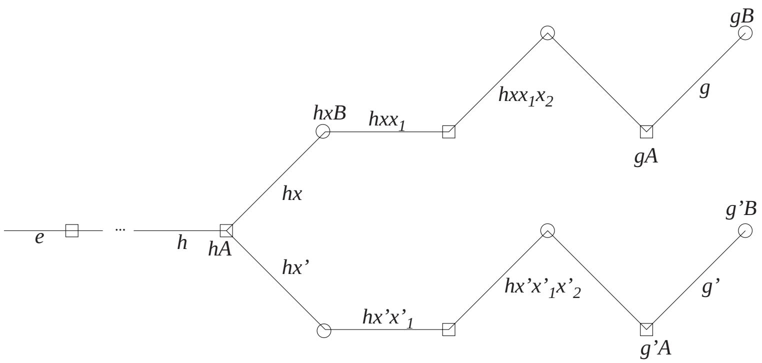
For the support property let $\hat{\varepsilon} > 0$ be given and obtain $R > 0$ such that conditions (2) hold for both α and β. Let $g,\, g' \in \Gamma$ be such that $d_\Gamma(g, g') \geqslant 2NR$. We show that

$$\langle \xi_g, \xi_{g'} \rangle = \langle \gamma_g, \gamma_{g'} \rangle \langle \tau_g, \tau_{g'} \rangle$$

has absolute value less than $\hat{\varepsilon}$. Each of the terms in this product has absolute value not greater than 1. Also, according to the lemma if $d_T(g,g')\geqslant 2N$ we have $\langle \tau_g, \tau_{g'}\rangle = 0$. Therefore, it suffices to show that $\langle \gamma_g, \gamma_{g'}\rangle$ has absolute value less than $\hat{\varepsilon}$ under the assumption that $d_T(g,g')<2N$. In order to show this, write $g = hxx_1\dots x_n$ and $g' = hx'x'_1\dots x'_m$ as products without cancellation as in (1). With the convention that empty products are 1 we have

$$\langle \gamma_g, \gamma_{g'}\rangle = \left(\prod_1^n \langle \gamma_{x_i}, \omega\rangle\right)\langle \gamma_x, \gamma_{x'}\rangle\left(\prod_1^m \langle \omega, \gamma_{x'_j}\rangle\right), \tag{4}$$

where γ's on the right-hand side stand for α's and β's as appropriate and consistent with expressions (1). (Again, in the degenerate case $h = x = x' = e$ the middle term does not appear.) Indeed, assume for definiteness that x, $x'\in A$ and that x_n, $x'_m\in A$ and consider the following portion of the Bass–Serre tree:



Bearing in mind their recursive descriptions, we see that the components of γ_g and $\gamma_{g'}$ agree for all vertices other than those on the paths from hA to gA and to $g'A$. The components at the vertex hA are α_x and $\alpha_{x'}$, respectively, and contribute the term $\langle \gamma_x, \gamma_{x'}\rangle$. The components at the vertex hxB are β_{x_1} and ω, respectively, and contribute the term $\langle \gamma_{x_1}, \omega\rangle$. The other vertices are similar.

Now, each term in product (4) has absolute value not greater than 1 and we conclude by showing that at least one term has absolute value less than $\hat{\varepsilon}$. Assume this is not the case. Then each of the $d_T(g,g')$ non-zero terms in the expression (2) for $d_\Gamma(g,g')$ would necessarily be less than R and we would have

$$d_\Gamma(g,g')\leqslant d_T(g,g')R<2NR,$$

a contradiction.

For the convergence property let g, $g'\in\Gamma$ be such that $d_\Gamma(g,g')\leqslant C$. Observe that

$$\|\xi_g - \xi_{g'}\|\leqslant\|\gamma_g - \gamma_{g'}\| + \|\tau_g - \tau_{g'}\|.$$

Again write g and g' as products without cancellation as in (1) and consider expression (2) for $d_\Gamma(g,g')$. Recalling that our length functions and distances are integer valued, the number $d_T(g,g')$ of non-zero terms in (2) is not greater than C. According to the lemma we obtain $||\tau_g - \tau_{g'}|| < \varepsilon/2$. Further, arguing as above, and with the convention that empty sums are 0, we obtain

$$||\gamma_g - \gamma_{g'}|| \leqslant \left(\sum_1^n ||\gamma_{z_i} - \omega||\right) + ||\gamma_x - \gamma_{x'}|| + \left(\sum_1^m ||\gamma_{z_i'} - \omega||\right).$$

(Again, when $h = x = x' = e$ the middle term does not appear.) Since no term in expression (2) for $d_\Gamma(g,g')$ is greater than C each of these $\leqslant C$ terms is less than $\varepsilon/2C$ and $||\gamma_g - \gamma_{g'}|| \leqslant \varepsilon/2$.

The proof in the case of exactness is largely the same, the only difference being with the support condition. Essentially, instead of showing that product (4) is less than $\hat{\varepsilon}$ we are to show it is 0. However, with the strengthened hypothesis that $\langle \alpha_a, \alpha_{a'} \rangle = 0$, if $\ell_A(a^{-1}a') \geqslant R$, and similarly for β, the proof proceeds as for uniform embeddability. $\square$

Remark on non-trivial amalgamation. All the methods in this note can be used to prove corresponding results when the free product is replaced by the amalgamated free product of A and B over a *finite* subgroup G. Indeed, we replace ℓ_A and ℓ_B by new length functions $\tilde{\ell}_A$ on A and $\tilde{\ell}_B$ on B such that

(1) $\tilde{\ell}_A(g) = 0$, for every $g \in G$;
(2) $\tilde{\ell}_A$ is equivalent to ℓ_A; that is, there exist positive constants c and c' and a non-negative constant d such that for every $a \in A$,

$$c\tilde{\ell}_A(a) - d \leqslant \ell_A(a) \leqslant c'\tilde{\ell}_A(a) + d;$$

(3) $\tilde{\ell}_A(gah) = \tilde{\ell}_A(a)$, for all $a \in A$, and $g, h \in G$.

Similar properties are required of $\tilde{\ell}_B$. A proof of existence of such length functions can be found in Jolissaint [Jol90]. Having done so, the methods above apply to the free product of the *metric spaces* A/G and B/G, which is quasi-isometric to $A *_G B$.

The case of *arbitrary* amalgamation is somewhat more difficult and is treated by Dadarlat–Guentner [DG02].

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